Physics 618 2020

- Fourier duality & Pontryagin Duality
- Orthog. Rel's for matrix elements of irred. reps.
- Heisenberg Groups with no canonical Lagrangian subgroups.
- Induced Representations

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Last time: Uniqueness of SvN rep of $\mathrm{Heis}(S \times \hat{S})$. $V = L^2(S)$

$$(T_s \psi)(s') = \psi(s+s') \qquad s \in S$$

Relation to Fourier analy.

$$(M_\chi \psi)(s') = \chi(s')\psi(s') \qquad \chi \in \hat{S}$$

But $\hat{\hat{S}} \cong S$ so

$$\underline{\mathrm{Heis}(S \times \hat{S})} \cong \underline{\mathrm{Heis}(\hat{S} \times \hat{\hat{S}})}$$

So there must be an intertwiner

$$\mathcal{S} : \underline{L^2(S)} \longrightarrow \underline{L^2(\hat{S})}$$

$$\mathcal{S}\, T_s\, \mathcal{S}^{-1} = \hat{M}_s$$

$$\mathcal{S}\, M_\chi\, \mathcal{S}^{-1} = \hat{T}_\chi$$

Fourier transform $\mathcal{S} : \underset{\text{wvfn on } S}{\underline{\underline{\psi}}} \longmapsto \underset{\text{wvfn on } \hat{S}}{\underline{\underline{\hat{\psi}(\chi)}}} := \int_S \chi(s)^* \psi(s)\, ds$

e.g. if $S = \mathbb{R}$, $\hat{S} = \mathbb{R}$

$$\hat{\psi}(k) = \int_S e^{-ik\cdot x}\, \psi(x)\, dx$$

Recall

$$1 \to U(1) \to \mathrm{Heis}(S\times\hat{S}) \to S\times\hat{S} \to 1$$

$$k\big((s_1,\chi_1),(s_2,\chi_2)\big) = \frac{\chi_2(s_1)}{\chi_1(s_2)}$$

$$Z\big(\mathrm{Heis}(S\times\hat{S})\big) \cong U(1).$$

$$\begin{aligned} S\times\hat{S} &\cong \hat{S}\times S \\ &\underset{\text{P.D.}}{\cong} \hat{S}\times\hat{\hat{S}} \\ &= \underline{\hat{S}}\times\widehat{(\hat{S})} \rightsquigarrow \underline{\underline{L^2(\hat{S})}} \end{aligned}$$

$$S\times\hat{S} \rightsquigarrow L^2(S)$$

In fact, this is an isometry:

$$\langle \psi_1, \psi_2 \rangle_{L^2(S)} = \langle \hat{\psi}_1, \hat{\psi}_2 \rangle_{L^2(\hat{S})}$$

$\Longleftrightarrow$

$$\boxed{\begin{aligned} \int_{\hat{S}} \chi(s_1)^* \chi(s_2)\, d\chi &= \delta_S(s_1 - s_2) \\ \int_S \chi_1^*(s)\, \chi_2(s)\, ds &= \delta_{\hat{S}}(\chi_1 - \chi_2) \end{aligned}}$$

<u>Example</u>: $S = \Gamma \subset \mathbb{R}^n$

Pont. Dual $(\Gamma) \cong \mathbb{R}^n / \Gamma^\vee$

$\Longrightarrow$ Poisson Summation Formula

Demand: $\hat{S}: \chi \longrightarrow \chi(s)$ is continuous

This puts a topology on $\hat{S}$.

Orthogonality Relations for Matrix Elements of irreps of more general groups.

G: Can be <u>any</u> compact group

~~(and more generally large classes of noncompact groups e.g. loc. cpt. Abelian)~~

$\widehat{G} :=$ Set of all irreducible unitary f.d. reps of G / equivalence $= \mathrm{IRREP}(G)$.

$\widehat{G}$ is NOT A GROUP IN GENERAL.

Consider two irreps:

$\mathbb{C}$-vs.
$$\begin{cases} V_\mu, \ \rho_\mu : G \longrightarrow GL(V_\mu) & n_\mu = \dim_{\mathbb{C}} V_\mu \\ V_\nu, \ \rho_\nu : G \longrightarrow GL(V_\nu) & n_\nu = \dim_{\mathbb{C}} V_\nu \end{cases}$$

$$A : \underline{V_\nu} \longrightarrow \underline{V_\mu} \qquad \mathbb{C}\text{- linear tmn.}$$

$$\underline{A \in \mathrm{Hom}(V_\nu, V_\mu)}$$

f.d. $\mathbb{C}$plx unitary

μ: label distinguishing the irreps of G

"Irrep" means $\rho : G \to GL(V)$

If $W \subset V$ is a linear subspace s.t. $\rho(g) W \subset W$ for all W then either

$W = \{0\}$ or $W = V$

Two rep's are <u>equivalent</u> if $\exists$ invertible intertwiner

(V, ρ) (V', ρ')

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & V' \arrow[d, "\rho'(g)"] \\
V \arrow[r, "T"'] & V'
\end{tikzcd} \quad \forall g$$

<u>AND</u> T is invertible then they are equivalent.

For matrix rep's:

$$\forall g \quad \rho'(g) = S\,\rho(g)\,S^{-1}$$

for some $S \in GL(n, \mathbb{C})$

$$\mathrm{IRREP}(G) = \left\{ \begin{array}{l} \text{All unitary f.d.} \\ \text{cplx rep's of } G \end{array} \right\} \Big/ \underset{\text{equiv.}}{\sim}$$

label the distinct irreps of G by V_μ

define

$$\underset{V_\mu \to V_\mu}{} \quad \underset{V_\nu \to V_\mu}{} \quad \underset{V_\nu \to V_\nu}{}$$

$$\tilde{A} := \int_G \rho_\mu(g)\, A\, \rho_\nu(g^{-1})\, dg \leftarrow$$

Claim: $\tilde{A}$ is an <u>intertwiner</u>

i.e. an equivariant map.

Need to prove $\boxed{\rho_\mu(g_0)\tilde{A} = \tilde{A}\rho_\nu(g_0)}$

$$\tilde{A}\rho_\nu(g_0) = \int_G \rho_\mu(g)\, A\, \rho_\nu(g^{-1}g_0)\, dg$$

$\boxed{g \to g_0 g}$
$$= \int_G \rho_\mu(g_0 g)\, A\, \rho_\nu(g^{-1})\, \underline{\underline{d(g_0 g)}}$$

Left invce of measure $\to$ $\underset{\parallel}{}$ dg

$$= \int_G \rho_\mu(g_0 g)\, A\, \rho_\nu(g^{-1})\, dg$$

$$\downarrow \quad = \rho_\mu(g_0)\tilde{A} \quad \checkmark$$

$$\boxed{\tilde{A} = \delta_{\mu,\nu} \cdot \mathbb{1} \cdot \underline{\underline{\lambda_A}}}$$

by Schur's lemma

The two irreps have to be the same.

λ_A is a scalar depending on A

To determine it set $\mu = \nu$ and choose an ordered basis for V_μ so the $\rho(g)$ become $n_\mu \times n_\mu$ complex matrices. Set $\underline{\underline{A = e_{ij}}}$

$$\lambda_{ij}\, \delta_{kl} = \int_G \rho(g)_{ki}\, \rho(g^{-1})_{jl}\, dg$$

Set $k = l$ and sum:

$$\lambda_{ij}\, n_\mu = \int_G \underbrace{\rho(g^{-1})_{jk}\, \rho(g)_{ki}}_{\rho(1)_{ji} = \delta_{ij}}\, dg$$

$$\lambda_{ij}\, n_\mu = \delta_{ij} \cdot \underbrace{\left(\int_G dg \right)}_{\text{WLOG } = 1} \leftarrow \text{using } G \text{ compact.}$$

$$\lambda_{ij} = \frac{\delta_{ij}}{n_\mu}$$

Conclude:

$$\boxed{\int_G \rho_\mu(g)_{ki}\, \rho_\nu(g^{-1})_{jl}\, dg = \delta_{\mu\nu} \frac{\delta_{ij}\delta_{kl}}{n_\mu}}$$

These are the orthogonality relations on matrix elements of irreps.

Remark: Note that we did NOT use unitarity above! Above is true for any irred. cplx matrix reps of G.

But, for G compact we can alway change the inner product on the group so that, relative to an ON basis for that inner product $\rho(g)_{ij}$ are unitary matrices.

(V, ρ) <u>any</u> finite diml $\mathbb{C}$-rep. of G. (important: G is compact)

Choose <u>any</u> nondeg. inner product $\langle \psi_1, \psi_2 \rangle$.

define a new inner product:

$$\langle\langle \psi_1, \psi_2 \rangle\rangle := \int_G \langle \rho(g)\psi_1, \rho(g)\psi_2 \rangle \, dg$$

Still nondegenerate.

Unitarity

$$\langle\langle \rho(g)\psi_1, \rho(g)\psi_2 \rangle\rangle$$

$$\doteq \langle\langle \psi_1, \psi_2 \rangle\rangle \quad \text{any } g.$$

Then $\rho(g)$ are unitary w.r.t. the new inner product. $\blacksquare$

For unitary reps the orthog. relations look very beautiful.

$$\phi_\mu^{ij} \in L^2(G) = \left\{ \psi : G \to \mathbb{C} \;\middle|\; \int_G \psi^* \psi \, dg < \infty \right\}$$

$$\phi_\mu^{ij} : \quad g \longmapsto \sqrt{n_\mu}\, \big(\rho_\mu(g)\big)_{ij}$$

<u>Orthog. Rel's</u> $\quad \phi_\mu^{ij}$ is an ON basis for $L^2(G)$!

$$\boxed{\int_G \left(\phi_\mu^{ij}(g)\right)^* \phi_\nu^{kl}(g)\, dg = \delta_{\mu\nu}\, \delta_{ik}\, \delta_{jl}}$$

For $G = SU(2)$

$\phi_\mu^{ij} \rightsquigarrow$ Wigner functions

$$D^j_{m,m'}(u)$$

Special cases: spherical haromonics, assoc. Legendre functions

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In general nice consequence

$L^2(G)$ is a $G \times G$ repn.

$$\Big((g_L, g_R)\cdot\psi\Big)(g)$$

$$:= \psi(g_L^{-1} g g_R)$$

As a $G \times G$ repn it is completely reducible

$$L^2(G) \cong \bigoplus_{\mu} \mathrm{End}(V_\mu)$$

$$= \bigoplus_{\mu} \overline{V_\mu} \otimes V_\mu$$

Corollary G - finite group.
take dimensions:

$$|G| = \sum_{\mu}' n_\mu^2$$

+ much much more!

Peter-Weyl theorem.

When working with reps it is often useful to work with

<u>Characters</u>:

$$\chi_V(g) := \mathrm{Tr}_V(\rho(g))$$

basis indpt., $\chi_V(hgh^{-1}) = \chi_V(g)$

$$\chi_\mu = \text{Character of } V_\mu$$

$$\boxed{\int_G \chi_\mu^*(g)\chi_\nu(g)\,dg = \delta_{\mu,\nu}}$$

Back to Heisenberg Groups

Other examples beyond $\mathrm{Heis}(S \times \hat{S})$

<u>Example</u>: R – commutative ring

$(R = \mathbb{Z}_n, \mathbb{Z})$

$$\left\{ M(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad a,b,c \in R \right\}$$

$$M(a,b,c)\,M(a',b',c')$$

$$= M(a+a', b+b', c+c'+ab')$$

$$\pi : M(a,b,c) \longrightarrow (a,b)$$

group homomorphism

$$z: \mathbb{R} \longrightarrow \{ M(a,b,c) \}$$

$$c \longmapsto M(0,0,c)$$

$$0 \to \mathbb{R} \to \mathrm{Heis}(\mathbb{R}\times\mathbb{R}) \longrightarrow \mathbb{R}\oplus\mathbb{R} \to 0$$

$$f((a,b),(a',b')) = ab' \quad \text{(*)}$$

with Abelian groups in additive notation the cocycle relation is

$$f(v_1, v_2) + f(v_1+v_2, v_3)$$

$$= f(v_1, v_2+v_3) + f(v_2, v_3)$$

Commutator function for (*)

$$k((a,b),(a',b')) = ab' - a'b$$

Specialize to $R = \mathbb{Z}/n\mathbb{Z}$

recover the finite Heisenberg group.

$$U^n = 1 = V^n$$

$$UV = q\,VU \qquad q^n = 1$$

q central.

<u>Example 2:</u> Clifford algebras and Extra-special groups.

work over $\mathbb{C}$

$$\boxed{\{\gamma_i, \gamma_j\} = 2\delta_{ij}} \qquad 1 \le i,j \le n.$$

<u>smallest</u> matrix reps of these relations.

<u>$n=2$</u> $\qquad \gamma_1 = \sigma^1, \ \gamma_2 = \sigma^2$

<u>$n=3$</u> $\qquad \gamma_1 = \sigma^1, \ \gamma_2 = \sigma^2, \ \gamma_3 = \sigma^3$

$n = 4$??

$$\left.\begin{aligned} \gamma_1 &= \sigma^1 \otimes \sigma^1 \\ \gamma_2 &= \sigma^1 \otimes \sigma^2 \\ \gamma_3 &= \sigma^1 \otimes \sigma^3 \\ \gamma_4 &= \sigma^2 \otimes \mathbb{1} \end{aligned}\right]$$

$A^2 = 1$

won't work
No B
anticommutes
w/ $\sigma^1, \sigma^2, \sigma^3$

$n = 5$: $\gamma_5 \sim \gamma_1 \cdots \gamma_4$

$\gamma_1, \dots \gamma_4$ as above

$\gamma_5 = \sigma^3 \otimes 1$

as in 4×4 rep of Cl_5

$n = 6$?

$$\begin{aligned} \gamma_1 &= \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \\ \gamma_2 &= \sigma^1 \otimes \sigma^1 \otimes \sigma^2 \\ \gamma_3 &= \sigma^1 \otimes \sigma^1 \otimes \sigma^3 \\ \gamma_4 &= \sigma^1 \otimes \sigma^2 \otimes 1 \\ \gamma_5 &= \sigma^1 \otimes \sigma^3 \otimes 1 \\ \gamma_6 &= \sigma^2 \otimes 1 \otimes 1 \end{aligned}$$

These are irred reps of Clifford. — $\mathbb{C}l_n$ = Cliff. algebra/$\mathbb{C}$ w/ gen's $e_1 \dots e_n$ rels $\{e_i, e_j\} = 2\delta_{ij}$

The dimension is
$$2^{[n/2]} \times 2^{[n/2]}$$

Irrep is almost unique.

$$\gamma_i \in \mathrm{Mat}_{d \times d}(\mathbb{C}) \qquad d = 2^{[n/2]}$$

$$\gamma_i \longrightarrow S \gamma_i S^{-1} \qquad S \in GL(d, \mathbb{C})$$

another rep. — equiv.

Also true that if we scale

$$\gamma_i \longrightarrow \epsilon_i \gamma_i \qquad \epsilon_i \in \{\pm 1\}$$

we also get a rep. of $\mathbb{C}l_n$, but it might not be equiv.

There might be no choice of S such that

$$S \gamma_i S^{-1} = \epsilon_i \gamma_i$$

Note that for $n=3$

$n=3$ $$\gamma_1 \gamma_2 \gamma_3 = i \mathbb{1}_{2\times 2}$$

$n=5$ $$\gamma_1 \cdots \gamma_5 = - \mathbb{1}_{4\times 4}$$

$$S \gamma_1 S^{-1} S \gamma_2 S^{-1} S \gamma_3 S^{-1}$$

$$= S \left(i \mathbb{1}_{2\times 2} \right) S^{-1} = \underline{i} \mathbb{1}_{2\times 2}$$

but suppose we chang $\gamma_i \to \underbrace{\epsilon_i \gamma_i}_{\tilde{\gamma}_i}$

with $\epsilon_1 \epsilon_2 \epsilon_3 = -1$

$$\tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 = - i \mathbb{1}_{2\times 2}$$

$\Rightarrow \exists$ At least two distinct irreps!

Thm: (a.) If n is even there is a unique irred rep. of $\mathbb{C}l_n$ of dimension $d = 2^{n/2}$

(b.) If n is odd there are exactly two irred reps of $\mathbb{C}l_n$ of dimension $d = 2^{(n-1)/2}$

They are distinguished by the sign of "volume form":

$$\gamma_1 \cdots \gamma_n$$

Notice that for n odd

$\gamma_1 \cdots \gamma_n$ is central, i.e. commutes with all the γ_i.

$w \in \mathbb{Z}_2^n \qquad \mathbb{Z}_2 = \{0,1\} \bmod 2$

$$\boxed{\gamma(w) = \gamma_1^{w_1} \cdots\cdots \gamma_n^{w_n}} \qquad \gamma(w)^2 = \pm 1$$

$(\gamma_i \gamma_j)^2 = -1$
$i \neq j$

typical element for $n = 7$.

$$\gamma(0110101) = \underbrace{\gamma_2 \gamma_3 \gamma_5 \gamma_7}$$

$$\gamma(w)\gamma(w') = \epsilon(w,w')\gamma(\underbrace{w+w'}_{\text{addition of binary codewords}})$$

$$\boxed{\mathcal{E}_n = \{\pm\gamma(w) \mid w \in \mathbb{Z}_2^n\}}$$

forms a group!!

$$1 \to \mathbb{Z}_2 \to \mathcal{E}_n \xrightarrow{\pi} \mathbb{Z}_2^n \to 0$$

$\mathbb{Z}_2 = \{\pm 1\}$ central

extension of $\mathbb{Z}_2^n$ by $\mathbb{Z}_2$.

Commutator function

$$k : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \longrightarrow \{\pm 1\}$$

$$k(\omega, \omega') = (-1)^{\sum_{i \neq j} \omega_i \omega_j'}$$

When is it nondegenerate?

If we choose $\gamma(\omega)$ ask:

Is there a $\gamma(\omega')$ that anticommutes?

Suppose $\sum_i \omega_i = 1 \mod 2$

$\exists\, i_0 \quad \omega_{i_0} = 0$ then γ_{i_0} anticommutes

$\sum_i \omega_i = 1 \mod 2$ $\omega_i = 1$ for all i ($\Rightarrow$ n is odd)

then all the γ_i commute

odd $\gamma_1 \cdots \gamma_n$: central $\Rightarrow$ k is degenerate

$\sum_i w_i = 0 \mod 2$ $\quad \exists\, i_0 \quad w_{i_0} \neq 0$

then γ_{i_0} anticommutes

e.g. $\quad \gamma_1 \gamma_2 \gamma_3 \gamma_4$

anticommutes with $\gamma_1, \gamma_2, \gamma_3, \gamma_4$

Conclusion : n is odd

do not get a Heisenberg extension because k is degen.

But if n is even we get a Heisenberg extension.

$$1 \to \mathbb{Z}_2 \to \mathcal{E}_{2n} \to \mathbb{Z}_2^{2n} \to 1$$

"Extra-special group 2_+^{2n+1}"

Important point:

There is no canonical presentation of $\mathbb{Z}_2^{2n}$ as a product of two maximal Lagrangian subgroups

I could choose $S \subset \mathbb{Z}_2^{2n}$ so that $\mathbb{Z}_2^{2n} \cong S \times \hat{S}$

But there are many different choices of S. None is distinguished.

How to give a representation of $\mathcal{E}$?

You could choose an S so that $\mathbb{Z}_2^{2n} \cong S \times \hat{S}$ and then use SvN rep.

Next time: Symplectic tori — see a similar phenomena.

How are the different choices of reps for different S's related?

Different choices of S are called "polarizations"

Like choosing coordinates and momenta.

Spin GROUP:

$\gamma_1 \cdots \gamma_n$

$v \in \mathbb{R}^n$ denote $\gamma \cdot v := \sum_{i=1}^{n} \gamma_i v_i$

(do not confuse with

$\gamma(w) = \gamma_1^{w_1} \cdots \gamma_n^{w_n} \quad w \in \mathbb{F}_2^n = \mathbb{Z}_2^n$)

Spin Group:

$$\mathrm{Spin}(n) = \left\{ \pm \underline{\underline{(\gamma \cdot v_1)}} \cdots \underline{\underline{(\gamma \cdot v_r)}} \;\middle|\; \begin{array}{l} r \text{ even and} \\ v_i^2 = 1 \\ 1 \le i \le r. \end{array} \right\}$$

$$\pi : \mathrm{Spin}(n) \longrightarrow O(n)$$

$\pi(\gamma(v)) \cdot w = w'$ defined

by

$$\gamma \cdot w' = -(\gamma \cdot v)(\gamma \cdot w)(\gamma \cdot v)^{-1}$$

ie $\pi(\gamma(v))$ = Reflection in hyperplane $\perp$ to v

Now for $n = 2m$ even.

There is a ! irrep of

$\mathbb{C}\ell_n$

But two inequivalent irreps

$\Delta^{\pm}$ of $Spin(n)$

$$P_{\pm} (\gamma \cdot v_1) \cdots (\gamma \cdot v_r) P_{\pm}$$

(r even)

$$P_{\pm} = \frac{1}{2}\left(1 \pm \xi \gamma_1 \cdots \gamma_{2n}\right)$$

ξ chosen so that

$$(\xi \gamma_1 \cdots \gamma_{2n})^2 = + \mathbb{1}.$$

$P_{\pm}$ proj. to $1/2$ dim

$$\dim_{\mathbb{C}} \Delta^{\pm} = 2^{m-1} = 2^{[\frac{n}{2}]-1}.$$